Homework 1 Solutions

Due: Thursday September 6th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Is the function

$$f : [0,1] \longrightarrow \mathbb{R}, \quad f(x) := \begin{cases} 1 & \text{if } x \in [\frac{1}{2^k}, \frac{1}{2^{k-1}}] \text{ for some odd } k \\ -1 & \text{if } x \in (\frac{1}{2^k}, \frac{1}{2^{k-1}}) \text{ for some even } k \\ 0 & \text{if } x = 0 \end{cases}$$

Riemann integrable?

Solution: Yes. Let $\epsilon > 0$. Choose n large enough so that $1/n < \epsilon/4$. Define

$$f_n : [1/n, 1] \longrightarrow \mathbb{R}, \quad f_n(x) := f(x)$$

Since f_n is continuous outside a finite number of points, it is Riemann integrable. Hence there exists a partition P_n of $[1/n, 1]$ so that $U(P_n, f_n) - L(P_n, f_n) < \epsilon/2$. Hence $U(\{0\} \cup P_n, f) - L(\{0\} \cup P_n, f) < 2/n + \epsilon/2 < \epsilon$. Hence f is Riemann integrable.

Problem 2: Let $f, g : [0,1] \longrightarrow \mathbb{R}$ be continuous functions. Define

$$F : [0,1] \longrightarrow \mathbb{R}, \quad F(x) := \begin{cases} f(x) & \text{if } x \text{ is rational} \\ g(x) & \text{otherwise.} \end{cases}$$

Compute the upper and lower Riemann integral of F in terms of integrals involving f and g.

Solution: Let $H := \max(f, g)$ and $h := \min(f, g)$. Then for any partition P of $[0,1]$, we have $U(P, F) = U(P, H)$ and $L(P, F) = L(P, h)$. Hence the upper integral is the Riemann integral of $\max(f, g)$ and the lower integral is the Riemann integral of $\min(f, g)$.

Problem 3: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function whose derivative f' is bounded and let $A \subset \mathbb{R}$ be a null set. Show that $f(A)$ is null.

Solution:

Since f is differentiable with bounded derivative, there is a constant C so that $|f'(x)| < C$ for all $x \in \mathbb{R}$. Let $I \subset \mathbb{R}$ be an interval and let $\overline{I} = [a, b]$ be its closure. Then $f(\overline{I})$ is an interval since f is continuous. Since f is continuous, $f|_{\overline{I}}$ attains its maximum and minimum on $\overline{I}$ and hence $f(\overline{I}) = [f(c), f(d)]$ for some $c, d \in I$. By the mean value theorem, there exists $e \in [a, b]$ so that

$$f(d) - f(c) = f'(e)(d - c) = |f'(e)||d - c| \leq |f'(e)||b - a| < C|b - a|.$$

Hence $l(f(I)) \leq l(\overline{I}) < C \cdot l(\overline{I}) = C \cdot l(I)$ for any interval $I \subset \mathbb{R}$.

For each $\epsilon > 0$, choose an interval covering $(I_n)_{n \in \mathbb{N}}$ of A so that $\sum_{n \in \mathbb{N}} l(I_n) < \epsilon/C$. Hence $(f(I_n))_{n \in \mathbb{N}}$ is an interval covering of $f(A)$ satisfying

$$\sum_{n \in \mathbb{N}} l(f(I_n)) < \sum_{n \in \mathbb{N}} C \cdot l(I_N) < \epsilon.$$

Hence $f(A)$ is null.

Problem 4: Let A, B be a null sets. Is it true that

$$A + B := \{a + b \ : \ a \in A, \ b \in B\}$$

is null? If it is then prove it, otherwise give a counterexample.

Solution: It is not true. Let A be the Cantor set and let B be the set of numbers in $[0,1]$ admitting a ternary (base 3) expansion which does not contain the digit 2. Then A is the set of numbers in $[0,1]$ admitting a ternary expansion not containing the digit 1.

The set B has measure 0 for the following reason: We have $B = \cap_{n\in\mathbb{N}} B_n$ where B_n is the set of numbers whose ternary expansion does not contain the digit 1 in its first $n+1$ digits. Since B_n is a union of 2^n intervals of length $(\frac{1}{3})^n$, we have $m^*(B_n) = (\frac{2}{3})^n$ and hence $m^*(B) \leq (\frac{2}{3})^n$ for all n and hence $m^*(B) = 0$. Hence B is null.

Now $[0,1] \subset A + B$ and hence $m^*(A+B) \geq m^*([0,1]) = 1$.